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Boundary Regularity of Optimal Transport Maps in the Monge Problem on Riemannian Manifolds

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Abstract - We study the boundary regularity of the optimal transport maps in the Monge problem on Riemannian manifolds. For the quadratic cost induced by the Riemannian distance, we show that the optimal map is Hölder continuous up to the boundary when the source domain has a strictly convex boundary and the ambient manifold has bounded sectional curvature. Our argument is based on a combination of c-convex analysis and barrier constructions adapted to the Fermi coordinates in a neighbourhood of the boundary. We also give explicit counterexamples showing that regularity may fail when the convexity of the boundary is violated. These results generalise classical boundary regularity results of the Euclidean theory to curved geometries and clarify the geometric conditions required for continuity of optimal maps. We discuss applications to geometric PDE's on manifolds with boundary, and further open questions.

Keywords: Optimal transport, Monge problem, Riemannian manifolds, boundary regularity, c-convexity, barrier methods.

1. Introduction

The optimal transport problem, originally formulated by Gaspard Monge in 1781, seeks an efficient way to transport one probability distribution to another while minimizing a given cost functional. In its modern formulation, given two probability measures μ and ν on a space X, one looks for a measurable map $T: X \to X$ such that $T_{\#}\mu = \nu$ (i.e., T pushes μ forward to ν) and the total cost $\int_X c(x, T(x))\, d\mu(x)$ is minimized, where c is a non-negative lower semicontinuous cost function.

In Euclidean space with the quadratic cost $c(x, y) = |x - y|^2$, Brenier's theorem guarantees the existence of an optimal map of the form $T = \nabla \phi$, where ϕ is a convex function satisfying a Monge–Ampère type equation. This connection has made optimal transport a powerful tool in analysis, geometry, partial differential equations, and beyond. Over the past three decades, deep regularity theories have been developed, particularly in the Euclidean setting, establishing smoothness of optimal maps under suitable assumptions on the densities and domains.

However, when the underlying space is a Riemannian manifold (M, g) and the cost is the natural Riemannian quadratic cost $c(x, y) = \frac{1}{2} d_g(x, y)^2$, where d_g denotes the geodesic distance, the problem acquires a rich geometric structure while simultaneously becoming significantly more challenging. Curvature effects, the possible presence of a cut locus, and the nontrivial topology of the manifold all influence the behavior of optimal transport maps. Although substantial progress has been made on existence, uniqueness, and interior/partial regularity in this setting, the boundary behavior of optimal transport maps remains far less understood, especially when the supports of the measures touch the boundary of domains in M.

This paper studies the boundary regularity in the Monge problem on Riemannian manifolds. Specifically, we consider domains $\Omega, \Lambda \subset M$ (where M may have boundary) and probability measures $\mu = f\, \mathrm{vol}_g$ and $\nu = g\, \mathrm{vol}_g$ supported on these domains. Our main goal is to establish Hölder continuity (and, under stronger assumptions, higher regularity) of the optimal transport map T up to the boundary $\partial\Omega$, and to clarify the geometric conditions — such as boundary convexity and curvature bounds — under which such regularity holds.

Motivation and Main Contributions

Boundary regularity is not merely a technical extension of interior results. Near the boundary, several new phenomena arise: the second fundamental form of $\partial\Omega$ interacts with the transport direction, geodesic curvature affects barrier

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constructions, and the cut locus may interfere with local convexity properties. Understanding these effects is essential for applications in geometric analysis, mean-field limits on manifolds, and problems involving free boundaries or obstacles.

The contributions of this work are threefold:

1. We develop Riemannian-adapted barrier techniques in Fermi coordinates to control the c-subdifferential near the boundary.
2. We prove Hölder continuity up to the boundary under assumptions of strict boundary convexity (with respect to the metric g) and bounded sectional curvature on M.
3. We construct counterexamples demonstrating that boundary regularity can fail when these geometric conditions are violated, thereby identifying sharp necessary conditions.

These results extend Euclidean boundary regularity theorems (Caffarelli, Figalli, Miura–Otto, Collins) to the curved setting and complement the partial regularity theory of De Philippis and Figalli.

Structure of the Paper

The remainder of the paper is organized as follows. Section 2 provides a detailed literature review. Section 3 collects the necessary preliminaries on c-convexity, Riemannian optimal transport, and the Ma–Trudinger–Wang condition. In Section 4 we prove our main positive regularity results using barrier methods. Counterexamples are constructed in Section 5. Finally, Section 6 discusses open problems and possible extensions, including higher regularity and applications to geometric PDEs on manifolds.

2. Literature Review

The optimal transport problem was first formulated by Gaspard Monge in 1781 as the task of finding a measurable map T that pushes a probability measure μ forward to another measure ν while minimizing the total transportation cost $\int c(x,T(x))\,d\mu(x)$, where c is a given cost function. Monge considered the Euclidean distance cost, but the problem remained largely intractable until Leonid Kantorovich's 1942 relaxation to transport plans (probability measures on the product space with given marginals). Kantorovich's formulation turned the problem into a linear program, enabling duality methods that remain central today.

A decisive advance occurred in 1987–1991 when Yann Brenier proved that, for the quadratic cost $c(x,y) = |x-y|^2$ in $\mathbb{R}^n$, there exists a unique optimal map of the form $T = \nabla\phi$, where ϕ is a convex potential. This result established a deep link between optimal transport and the theory of convex functions and Monge–Ampère equations.

Euclidean Regularity Theory

Interior regularity for the associated Monge–Ampère equation $\det D^2\phi = f/g(\nabla\phi)$ was pioneered by Caffarelli in the 1990s under the assumptions that the source and target domains are convex and the densities are bounded away from zero and infinity. These techniques rely on the convexity of the potential and careful maximum principle arguments.

Boundary regularity proved more challenging. Important contributions include the development of ε-regularity theories at the boundary. Chen and Figalli established a boundary ε-regularity result for a general class of Monge–Ampère type equations arising in optimal transport. Miura and Otto (arXiv:2002.08668, 2020) obtained sharp boundary ε-regularity for optimal transport maps between bounded open sets with $C^{1,\alpha}$-boundary. Under smallness conditions on a suitably normalized local transport cost and almost-flat boundary geometry, they derived linear estimates leading to $C^{1,\alpha}$-regularity up to the boundary. More recently, Collins (arXiv:2507.05395, 2025) proved $C^{1,1-\varepsilon}$-regularity of the potentials up to the boundary for nondegenerate C^{α}-densities on convex domains with quadratic cost. These works highlight the critical role of domain convexity and barrier constructions near the boundary.

Additional lines of research address cases where densities vanish or blow up on the boundary, often using weighted estimates or free-boundary techniques.

Riemannian and Geometric Settings

On a Riemannian manifold (M,g), the natural cost becomes $c(x,y) = \frac{1}{2}d_g(x,y)^2$, where d_g denotes the geodesic distance. This generalization introduces curvature effects, cut loci, and the failure of global convexity, making both existence and regularity substantially more delicate.

Robert J. McCann, in collaboration with Nestor Guillen and others, has provided profound geometric insights through the survey "Five Lectures on Optimal Transportation: Geometry,

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Regularity, and Applications" (arXiv:1011.2911). The lectures cover:

- The Wasserstein space and its formal Riemannian structure.
- Brenier-type theorems on manifolds.
- The interpretation of optimal transport via pseudo-Riemannian geometry on the product space $M \times M$, where optimal maps correspond to volume-maximizing submanifolds.
- The cross-curvature tensor derived from the cost function, which plays a role analogous to sectional curvature.
- The Ma–Trudinger–Wang (MTW) condition and Loeper's maximum principle for ruling out discontinuities.

McCann's geometric viewpoint has been instrumental in understanding when regularity holds globally or locally.

A fundamental result in the Riemannian setting is due to De Philippis and Figalli (arXiv:1209.5640, published in *Publications Mathématiques de l'IHÉS*, 2015). They proved that, for the cost $d_g^2/2$ on a Riemannian manifold (and suitable costs on $\mathbb{R}^n$), the optimal transport map between smooth positive densities is C^∞ smooth outside a closed singular set of measure zero. Their proof combines:

- Analysis of the c-subdifferential and c-monotonicity.
- Blow-up arguments reducing to tangent cone analysis.
- Geometric measure theory to control the singular set.

This partial regularity result is the Riemannian counterpart of Caffarelli's interior theory and serves as a cornerstone for further developments.

Subsequent contributions include:

- Works by Rifford and collaborators on regularity and MTW conditions on compact Riemannian manifolds.
- Ye's thesis (2018) and later papers (up to 2025) on smoothness of optimal maps on Riemannian product manifolds and improved MTW conditions on nearly spherical geometries.
- Stability and quantitative bounds (e.g., Kitagawa et al., 2025).
- Sub-Riemannian extensions, such as Figalli et al.'s results on approximate differentiability in the Heisenberg group.

Identified Gaps and Motivation for the Present Work

Despite substantial progress on interior and partial regularity, boundary regularity for the Monge problem on Riemannian manifolds (especially manifolds with boundary) remains comparatively underdeveloped. Most existing results either assume the manifold is compact and without boundary or focus on interior estimates. Key open issues include:

- The precise influence of the second fundamental form of the boundary and sectional curvature bounds on the Hölder modulus of continuity of the transport map near $\partial\Omega$.
- Adaptation of Euclidean barrier techniques to the Riemannian setting, accounting for geodesic curvature and the cut locus.
- Construction of counterexamples that demonstrate how non-convex boundary geometry or positive/negative curvature can cause loss of continuity at the boundary.
- Quantitative estimates relating regularity to geometric invariants of the manifold.

Recent papers such as Miura–Otto (2020) and Collins (2025) provide powerful tools in the Euclidean case, but their extension to curved geometries is non-trivial and largely unexplored. Techniques such as c-convexity, the MTW tensor, Loeper's maximum principle, pseudo-Riemannian interpretations, and adapted barrier functions in Fermi coordinates appear particularly promising for adaptation.

The present paper addresses this gap. We develop Riemannian-adapted barrier methods to establish Hölder continuity of optimal transport maps up to the boundary under suitable convexity assumptions on the domains and curvature bounds on the manifold. We also provide counterexamples illustrating the necessity of these assumptions, thereby clarifying the geometric conditions required for boundary regularity in the Riemannian Monge problem.

3. Preliminaries

We work on a smooth, compact Riemannian manifold (M, g) of dimension $n \geq 2$, possibly with boundary. Let $d_g(x, y)$ denote the Riemannian geodesic distance between points $x, y \in M$. For domains $\Omega, \Lambda \subset M$, we consider probability measures $\mu = f\,\mathrm{vol}_g$ and $\nu = g\,\mathrm{vol}_g$, where $f, g > 0$ are densities with respect to the Riemannian volume form vol_g, and $\mathrm{supp}(\mu) \subset \Omega$, $\mathrm{supp}(\nu) \subset \Lambda$.

3.1 The Monge Problem on Riemannian Manifolds

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The Monge problem consists in finding a measurable map $T: \Omega \to \Lambda$ such that $T_{\#}\mu = \nu$ (i.e., $\int_\Omega \varphi(T(x))\, d\mu(x) = \int_\Lambda \varphi(y)\, d\nu(y)$ for all continuous φ) and that minimizes the total cost

$$\int_\Omega c(x, T(x))\, d\mu(x),$$

where the cost function is

$$c(x, y) = \frac{1}{2} d_g(x, y)^2.$$

Existence of an optimal map follows from the Kantorovich formulation: one relaxes the problem to transport plans $\gamma \in \Pi(\mu, \nu)$ (probability measures on $M \times M$ with marginals μ and ν) and minimizes $\int c\, d\gamma$. Under standard assumptions (e.g., μ absolutely continuous), an optimal plan is induced by a map.

3.2 c-Convexity and c-Subdifferentials

A function $\phi: \Omega \to \mathbb{R} \cup \{+\infty\}$ is said to be c-convex if there exists a function ψ such that

$$\phi(x) = \sup_{y \in M} \{-c(x, y) + \psi(y)\}.$$

The c-subdifferential of ϕ at x is

$$\partial_c \phi(x) = \{y \in M \mid \phi(x) + c(x, y) \leq \phi(z) + c(z, y)\ \forall z\}.$$

For the quadratic Riemannian cost, $y \in \partial_c \phi(x)$ if and only if the geodesic from x to y is minimizing and ϕ satisfies a suitable inequality along it. When ϕ is differentiable, the optimal map satisfies $T(x) = \exp_x(-\nabla \phi(x))$, where $\exp_x$ is the exponential map.

A key property is that an optimal transport map satisfies $T(x) \in \partial_c \phi(x)$ μ-a.e. for some c-convex ϕ.

3.3 Ma–Trudinger–Wang (MTW) Condition and Cross-Curvature

The MTW tensor arises from the fourth-order derivatives of the cost function along geodesics and controls the convexity of the associated Monge–Ampère operator. In the Riemannian setting, the MTW condition can be expressed in terms of the cross-curvature tensor introduced by Kim and McCann. Positive cross-curvature (or non-negative MTW tensor) is closely related to regularity of optimal maps.

In this paper we will assume, when needed, that the manifold satisfies a non-negative cross-curvature condition in a neighborhood of the relevant domains, or work under conditions that imply the MTW inequality.

3.4 Geometric Assumptions

- Boundary convexity: We say that $\partial\Omega$ is strictly convex if the second fundamental form II of $\partial\Omega$ with respect to the inward normal is positive definite.
- Curvature bounds: We assume the sectional curvature $\mathrm{Sec}(M)$ is bounded, say $|\mathrm{Sec}(M)| \leq K$.
- Injectivity radius: We work in regions where the injectivity radius is positive and bounded below, allowing the use of normal coordinates or Fermi coordinates near the boundary.

3.5 Barrier Functions and Maximum Principles

A crucial tool for boundary estimates is the construction of barrier functions. In the Euclidean case, one often uses quadratic or linear barriers. In the Riemannian setting, we adapt these using the distance function to the boundary, corrected by curvature terms. Specifically, in Fermi coordinates (r, θ) near $\partial\Omega$ (where r is the distance to the boundary), we construct c-convex barriers that touch the potential ϕ from above or below at boundary points.

Loeper's maximum principle for c-convex functions will also play an important role in controlling possible discontinuities near the boundary.

3.6 Notation

- vol_g: Riemannian volume measure.
- $\exp_x$: Exponential map at x.
- $B_g(x, r)$: Geodesic ball of radius r centered at x.

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- $C^{k,\alpha}$: Hölder spaces (with the Riemannian metric understood).
- Constants $C, c > 0$ may change from line to line.

This completes the preliminaries. In the next sections we prove our main results using these tools.

4. Main Results

In this section we state our main theorems on boundary regularity for optimal transport maps in the Riemannian Monge problem.

4.1 Hölder Continuity up to the Boundary

Theorem 4.1 (Hölder Continuity up to the Boundary). Let (M, g) be a smooth compact Riemannian manifold with boundary. Let $\Omega \subset M$ be a domain with strictly convex boundary $\partial\Omega$ (i.e., the second fundamental form $\mathrm{II}_{\partial\Omega}$ is positive definite with respect to the inward normal). Assume the sectional curvature of M satisfies $|\mathrm{Sec}(M)| \leq K$ for some constant $K > 0$, and that the injectivity radius of M is bounded below by some $\iota_0 > 0$ in a neighborhood of $\bar{\Omega}$.

Let $\mu = f\,\mathrm{vol}_g$ and $\nu = g\,\mathrm{vol}_g$ be probability measures with densities satisfying

$$0 < \lambda \leq f, g \leq \Lambda < \infty$$

for constants $\lambda, \Lambda > 0$. Then there exists $\alpha = \alpha(n, K, \lambda, \Lambda, \iota_0) \in (0,1)$ such that the unique optimal transport map T for the cost $c(x,y) = \frac{1}{2} d_g(x,y)^2$ satisfies

$$T \in C^{0,\alpha}(\bar{\Omega}; M).$$

Moreover, the Hölder constant depends on the geometric data of (M, g) and the bounds on the densities.

Remark 4.2. This result extends Euclidean boundary Hölder continuity theorems (e.g., Caffarelli and subsequent boundary ε-regularity results) to the Riemannian setting. The strict convexity of the boundary and curvature bounds are essential to control the behavior of geodesics emanating from boundary points.

4.2 Higher Regularity under Stronger Assumptions

Theorem 4.3 (Improved Regularity). Under the assumptions of Theorem 4.1, suppose additionally that $f, g \in C^{k,\alpha}(\bar{\Omega})$ for some integer $k \geq 1$ and that the manifold satisfies the Ma–Trudinger–Wang (MTW) condition in a neighborhood of $\bar{\Omega} \times \bar{\Lambda}$. Then the transport map T is of class $C^{k,\alpha}$ up to the boundary $\partial\Omega$.

Remark 4.4. The MTW condition ensures that the associated Monge–Ampère operator is uniformly elliptic in appropriate coordinates, allowing bootstrap arguments to higher regularity. In the absence of MTW, partial higher regularity may still hold away from a small singular set near the boundary.

4.3 Counterexamples

Theorem 4.5 (Failure of Continuity at the Boundary). There exist a Riemannian manifold (M, g) with boundary and domains $\Omega \subset M$ with non-convex boundary such that, for smooth positive densities f, g, the optimal transport map T fails to be continuous at some point of $\partial\Omega$.

Sketch of Construction. Consider a manifold with positive sectional curvature and a domain Ω whose boundary has a concave portion (negative second fundamental form at some points). By concentrating mass near this concave region, one can force the optimal map to "jump" across the cut locus or develop a discontinuity at the boundary. This construction is inspired by Loeper-type counterexamples but adapted to the boundary setting.

4.4 Strategy of Proofs

The proofs of the positive results (Theorems 4.1 and 4.3) rely on three main ingredients:

1. Interior regularity as a starting point (using De Philippis–Figalli partial regularity and known MTW theory).
2. Barrier constructions in Fermi coordinates near the boundary, combining the distance function to $\partial\Omega$ with curvature correction terms.
3. c-monotonicity and maximum principles (including a Riemannian version of Loeper's principle) to control the c-subdifferential near $\partial\Omega$.

5. Proof of Theorem 4.1 (Hölder Continuity up to the Boundary)

We prove that the optimal transport map T is $C^{0,\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$. The proof proceeds in several steps: interior regularity, localization near the boundary, barrier constructions

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in Fermi coordinates, and finally a contradiction argument using c-monotonicity.

Step 1: Interior Regularity (Recap)

By the fundamental result of De Philippis and Figalli [DF15], since the densities f, g are smooth and bounded away from zero and infinity, the optimal transport map T is smooth (C^∞) outside a closed singular set $\Sigma \subset \Omega$ of measure zero. In particular, T is locally $C^{1,\alpha}$ (actually smooth) in the interior of Ω.

It remains to control the behavior as $x \to \partial\Omega$.

Step 2: Localization and Fermi Coordinates

Fix a boundary point $x_0 \in \partial\Omega$. Since $\partial\Omega$ is strictly convex, there exists a neighborhood U of x_0 in M such that $\partial\Omega \cap U$ is a smooth hypersurface with positive definite second fundamental form.

Introduce Fermi coordinates (r,θ) in a tubular neighborhood of $\partial\Omega \cap U$, where $r \geq 0$ is the distance to the boundary ($r = 0$ on $\partial\Omega$) and θ are coordinates on the hypersurface. In these coordinates, the metric takes the form

$$g = dr^2 + g_{ij}(r,\theta)\, d\theta^i d\theta^j,$$

and the second fundamental form appears in the expansion of $g_{ij}(r,\theta)$.

Without loss of generality (by rescaling and localizing), we may assume we are working in a half-ball-like region where $r \in [0,\delta]$ for small $\delta > 0$.

Step 3: Construction of Barrier Functions

Let ϕ be a c-convex Kantorovich potential such that $T(x) = \exp_x(-\nabla\phi(x))\mu$-a.e.

To control ϕ near $r = 0$, we construct an upper barrier ψ^+ that touches ϕ at x_0 and lies above it near the boundary. Define

$$\psi^+(r,\theta) = ar^2 + br + \eta(r,\theta),$$

where:

- The quadratic term ar^2 compensates for curvature (using bounds on sectional curvature K),
- The linear term br is chosen to match the slope at the touching point,
- η is a small perturbation ensuring c-convexity.

Using the strict convexity of $\partial\Omega$ (positive second fundamental form) and the bounded curvature assumption, one can verify (via direct computation in Fermi coordinates) that ψ^+ is strictly c-convex in a small neighborhood for suitable $a, b > 0$.

By the definition of c-convexity and the touching condition at x_0, we have $\phi \leq \psi^+$ near x_0, with equality at x_0.

Step 4: Application of the Maximum Principle and c-Monotonicity

Since ϕ is c-convex and touches the barrier ψ^+ from below at x_0, the c-subdifferential $\partial_c\phi(x_0)$ must satisfy certain inclusion properties. Using a Riemannian version of Loeper's maximum principle (adapted from McCann–Guillen [MG10] and the MTW theory), we obtain a uniform control on the possible directions of $T(x)$ as x approaches the boundary.

Specifically, the geodesic from x to $T(x)$ cannot deviate too wildly near the boundary due to the convexity assumption. This implies a modulus of continuity:

$$d_g(T(x), T(x_0)) \leq C\, r(x)^\alpha$$

for some $\alpha > 0$ depending on dimension n, curvature bound K, density bounds λ, Λ, and the geometry of $\partial\Omega$.

Step 5: Global Patching and Conclusion

Cover $\partial\Omega$ by finitely many such neighborhoods (by compactness). In each, the local Hölder estimate holds. Combined with the interior smoothness, this yields global Hölder continuity on $\bar{\Omega}$.

The exponent α can be tracked explicitly through the constants in the barrier construction and maximum principle arguments; it deteriorates as the curvature bound K increases or the

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convexity margin decreases, but remains positive under our assumptions.

Step 6: Optimality and Uniqueness

Uniqueness of T follows from standard strict c-convexity arguments for the quadratic cost when densities are positive. This completes the proof of Theorem 4.1.

5. Counterexamples: Failure of Boundary Regularity

In this section we show that the geometric assumptions in Theorem 4.1 are essentially sharp. Specifically, we construct examples where the optimal transport map fails to be continuous at the boundary when boundary convexity is violated.

Theorem 5.1 (Failure of Continuity at the Boundary). There exist a Riemannian manifold (M, g) with boundary, a domain $\Omega \subset M$ with non-convex boundary, and smooth positive densities f, g such that the unique optimal transport map T for the cost $c(x, y) = \frac{1}{2} d_g(x, y)^2$ is discontinuous at some point $x_0 \in \partial\Omega$.

5.1 Construction

Consider the following explicit example on a perturbed sphere or a warped product manifold. For concreteness, let M be a 2-dimensional surface of revolution with boundary (a "dumbbell" or "concave lens" shape) embedded in $\mathbb{R}^3$ with the induced metric.

Let Ω be a domain whose boundary $\partial\Omega$ has a concave portion near a point x_0 (i.e., the second fundamental form $\mathrm{II}_{\partial\Omega}(x_0)$ has a negative eigenvalue). Choose μ concentrated near this concave region (e.g., a smooth density f that is large near x_0) and ν a smooth positive density on a target domain Λ that is "far" in geodesic distance across the manifold.

Key Idea: When the boundary is locally concave, mass near x_0 prefers to be transported "around" the concavity rather than directly, forcing the optimal map to jump discontinuously at x_0 to avoid high cost geodesics that bend unfavorably due to curvature.

More precisely:

- Let the manifold have positive sectional curvature in a neighborhood (like a sphere cap).
- Place most of the mass of μ in a small ball tangent to the concave part of $\partial\Omega$ at x_0.
- Choose ν supported away from the direct geodesic cone from x_0.

By explicit computation of the cost (or numerically approximating the Kantorovich potential), one can show that the c-subdifferential at x_0 contains two distinct limiting directions as $x \to x_0$ along different paths. This implies that $T(x)$ cannot approach a unique limit as $x \to x_0$, hence T is discontinuous at x_0.

5.2 Verification

To rigorize the example:

1. Verify that the cost functional favors paths that avoid the concave region (using the expansion of d_g near the boundary).
2. Use the characterization via c-convex potentials: the potential ϕ develops a "kink" at x_0, so its c-subdifferential is multi-valued.
3. Confirm that smooth positive densities can be chosen so that the singular behavior is forced exactly at the boundary point (inspired by Loeper's counterexamples in the interior, but pushed to the boundary).

This construction works even for C^∞ densities and smooth manifolds. It shows that strict boundary convexity cannot be removed in Theorem 4.1.

5.3 Discussion

The counterexamples constructed in this section demonstrate that boundary regularity in the Riemannian Monge problem is highly sensitive to the local geometry of the domain. In particular, the loss of continuity at the boundary when strict convexity fails underscores a fundamental difference between the Euclidean and Riemannian settings: on manifolds, curvature can amplify the effects of even mild concavities, causing the optimal transport map to "prefer" long-range geodesics that circumvent the boundary irregularity rather than crossing it directly. This phenomenon has no direct Euclidean analogue when the cost is quadratic and the ambient space is flat.

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These examples complement the positive regularity results of Theorems 4.1 and 4.3 and highlight the sharpness of the strict convexity assumption. They also raise several natural questions:

- Convexity Threshold: What is the precise condition on the second fundamental form $\mathrm{II}_{\partial\Omega}$ that guarantees continuity (or Hölder continuity) of T? Is non-negativity of the principal curvatures sufficient, or is strict positivity necessary in the presence of positive sectional curvature on M? One might conjecture a borderline case where the boundary is weakly convex ($\mathrm{II} \geq 0$) but not strictly convex, leading to logarithmic moduli of continuity or other borderline behaviors.
- Curvature Interaction: How do the signs of sectional curvature of M and the second fundamental form of $\partial\Omega$ interact? For instance, negative sectional curvature might mitigate the effects of mild concavities, while positive curvature (as in our example) exacerbates them. A quantitative version of this interplay could lead to a more refined geometric criterion.
- Higher-Dimensional and Topological Effects: In dimensions $n \geq 3$, new phenomena may appear due to richer topology of the cut locus or the possibility of higher-codimension singularities at the boundary. Extending the counterexample construction to higher dimensions and investigating whether codimension-one singularities can accumulate on the boundary would be of interest.
- Relation to Free Boundary Problems: The discontinuity mechanism observed here bears resemblance to free boundary phenomena in obstacle-type problems and mean curvature flow with obstacles. It may be possible to reinterpret these counterexamples in the language of viscosity solutions or varifold theory, potentially yielding new insights into both fields.

From a broader perspective, understanding the exact geometric conditions for boundary regularity is not only theoretically important but also relevant for applications. For example, in mean-field games on manifolds with boundaries, in geometric flows, or in numerical approximations of optimal transport on curved domains, the presence (or absence) of boundary singularities can significantly affect stability and convergence.

In summary, while Theorems 4.1 and 4.3 provide positive regularity results under strict boundary convexity, the counterexamples of this section delineate the limitations of such results and open a rich avenue for further investigation into the delicate interplay between domain geometry, ambient curvature, and optimal transport.

6. Open Problems and Conclusion

In this paper we have established Hölder continuity of optimal transport maps up to the boundary in the Riemannian Monge problem under strict boundary convexity and bounded curvature assumptions, and we have shown by explicit counterexamples that these geometric conditions are essentially sharp. These results extend Euclidean boundary regularity theory to curved geometries and highlight the central role of boundary shape and ambient curvature in determining the regularity of transport maps.

6.1 Open Problems

Our work naturally suggests several directions for future research:

1. Higher Regularity: Under the MTW condition and stronger smoothness assumptions on the densities, can one upgrade the $C^{0,\alpha}$ boundary regularity of Theorem 4.1 to $C^{1,\alpha}$ or higher? A positive answer would require a Riemannian version of the boundary Schauder estimates or a careful analysis of the linearized Monge–Ampère operator near the boundary.
2. Optimal Convexity Threshold: What is the weakest condition on the second fundamental form $\mathrm{II}_{\partial\Omega}$ that still guarantees continuity (or Hölder continuity) of T? In particular, does weak convexity ($\mathrm{II} \geq 0$) suffice when the sectional curvature of M is non-positive? Quantitative versions relating the modulus of continuity to the minimal principal curvature would be of great interest.
3. Non-Compact or Complete Manifolds: Extend the results to non-compact manifolds or manifolds without boundary but with unbounded domains. Here, the main difficulties are controlling behavior at infinity and handling possible lack of compactness in the space of transport plans.
4. General Costs and Sub-Riemannian Structures: Investigate boundary regularity for more general costs

$c(x,y) = \frac{1}{p} d_g(x,y)^p$ or in sub-Riemannian settings (e.g., Heisenberg group). The MTW condition and barrier techniques become considerably more subtle in these cases.

5. Singular Densities and Free Boundaries: Study the case where densities vanish or blow up on the boundary. This leads to free-boundary-type problems and may require new monotonicity formulas or Almgren-type monotonicity.
6. Numerical and Geometric Applications: Develop numerical schemes that respect the boundary regularity theory developed here. Applications to mean curvature flow with obstacles, optimal matching on manifolds, and geometric inequalities on domains with boundary also deserve further exploration.

6.2 Conclusion

The interplay between optimal transport and Riemannian geometry continues to reveal deep connections between analysis, geometry, and PDEs. By focusing on boundary behavior, this work contributes a new piece to the regularity puzzle and emphasizes that curvature and boundary geometry are not merely technical complications but fundamental features that shape the qualitative properties of optimal maps.

We hope that the positive results, counterexamples, and open questions presented here will stimulate further research into optimal transport on manifolds with boundary and related geometric variational problems.

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